Math 142 Lecture 22 Notes

Daniel Raban

April 10, 2018

1 Review: Free Products, Surfaces, and Orbit Spaces

1.1 Free products vs Cartesian products

What is the difference between $\mathbb{Z}^2$ and F_2? $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$, while $F_2 = \mathbb{Z} * \mathbb{Z}$. The difference is that in $\mathbb{Z}^2$, we assume that elements commute, while they do not in F_2. If we write out the presentations, we have

$$\mathbb{Z} = \langle a \rangle, \qquad F_2 = \langle a, b \rangle .$$

The elements of F_2 are $a, b, a^{-1}b, aba^2b^3a^{-7}b^4, \dots$.

$$\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle .$$

The elements of $\mathbb{Z}^2$ are $a, b, a^{-1}b, a^{-4}b^8, \dots$, noting now that a and b commute. So we have elements of the form $a^m b^n$ for $n, m \in \mathbb{Z}$.

Here is one of the practice problems for Midterm 2. It says to compute the Abelianization of the following group.

$$G = \left\langle a, b, c \mid ab^2a^{-1} = 1, ac^{-1} = 1 \right\rangle$$

The second relation says that $a = c$, so we can replace all instances of c by a.

$$= \left\langle a, b \mid ab^2a^{-1} = 1 \right\rangle$$

The remaining relation says that $ab^2 = a$, which then simplifies to $b^2 = 1$.

$$\begin{aligned} &= \left\langle a, b \mid b^2 = 1 \right\rangle \\ &= \underbrace{\mathbb{Z}}_{a} * \underbrace{\mathbb{Z}/2\mathbb{Z}}_{b} \end{aligned}$$

So then

$$\mathrm{Ab}(G) = \left\langle a, b \mid b^2 = 1, ab = ba \right\rangle \cong \mathbb{Z} \times \mathbb{Z}_2.$$

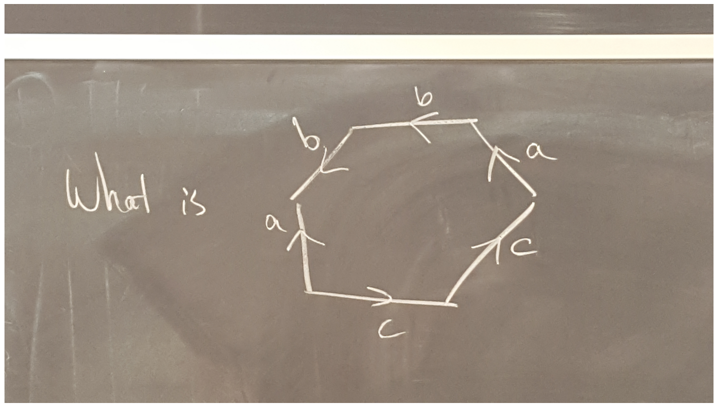
This generalizes to the following fact.

Theorem 1.1.

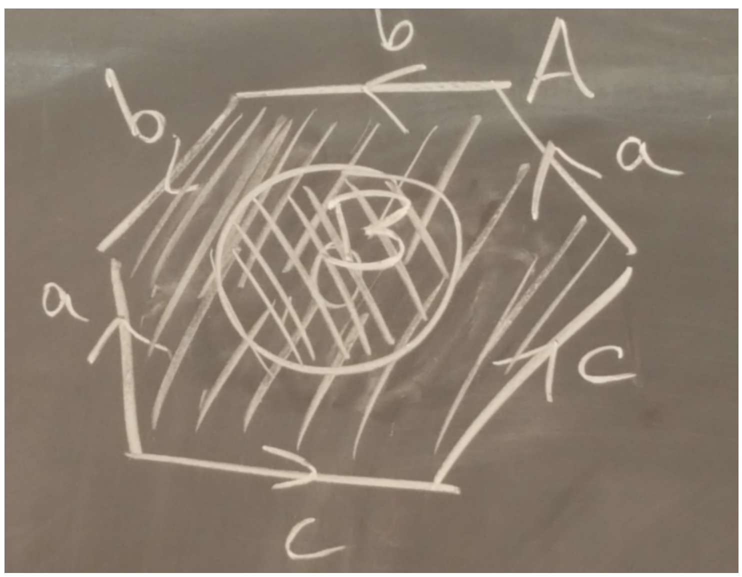
$$\mathrm{Ab}(G_1 * G_2) \cong \mathrm{Ab}(G_1) \times \mathrm{Ab}(G_2).$$

1.2 Recognizing surfaces using the Seifert-van Kampen theorem

How can we tell what a surface is given a cellular decomposition?



Our first way to do this is to use our lemmas about the word of a cellular decomposition. Another is to use the Seifert-van Kampen theorem to separate the surface into parts.

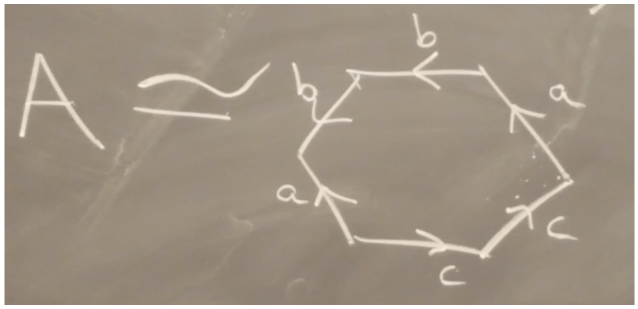


$$B \cong D^2 \simeq \text{point}$$

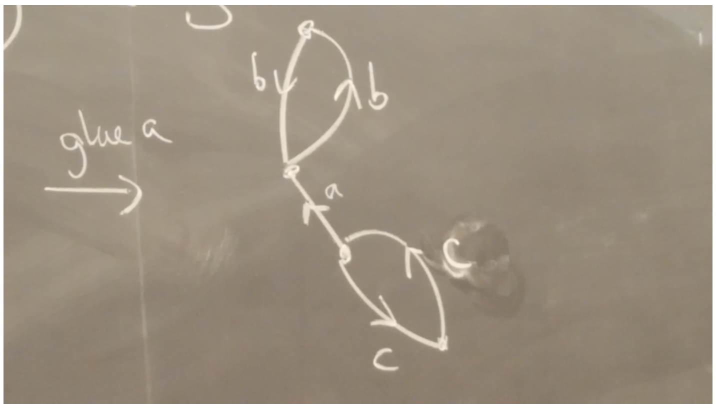
$$A \cap B \cong S^1 \times (0,1) \simeq S^1$$

$$A \simeq H,$$

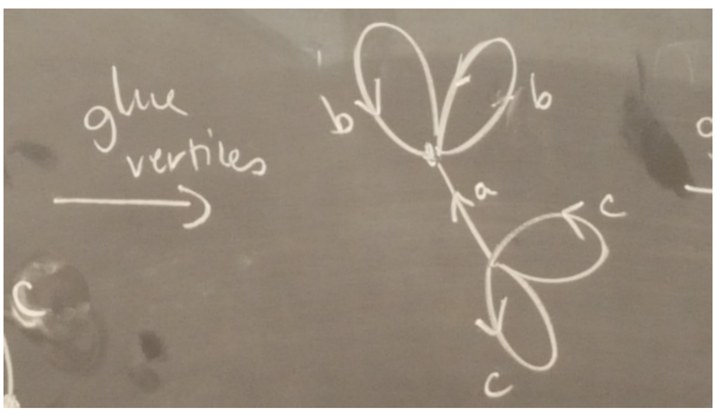
where H is the boundary of this hexagon.



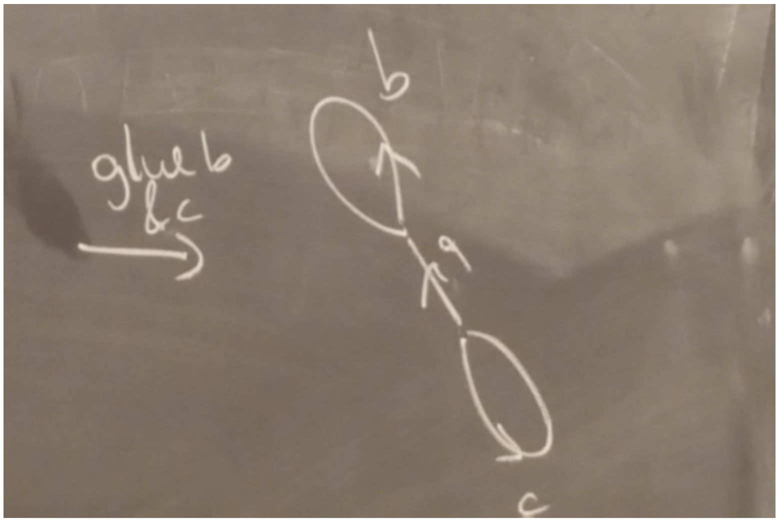
To figure out what H is homotopy equivalent to, glue the a sides together.



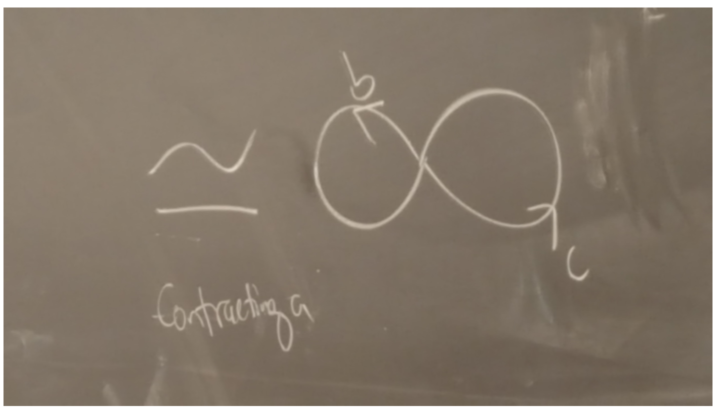
The two vertices of the b edges are the same, and the two vertices of the c edges are the same, so glue them together.



Then we can overlap the b loops together and the c loops together to get



Shortening the side labeled a gives us



So we get that

$$\pi_1(A \cap B) \cong \mathbb{Z} = \{[\gamma]^n : n \in \mathbb{Z}\}, \qquad \pi_1(A) \cong F_2 = \langle b, c \rangle, \qquad \pi_1(B) \cong 1.$$

If $i_1 : A \cap B \to A$ and $i_B : A \cap B \to B$ are the inclusion maps, then

$$\pi_1(X) \cong \frac{\pi_1(A) * \pi_1(B)}{N} \cong F_2/N = \langle b, c \mid (i_A)_*([\gamma]) = (i_B)_*([\gamma]) \rangle$$

Note that $i_A([\gamma])$ is going once around the border of the hexagon. So, looking at our pictures, we get that this is going around loop b twice and then loop c twice.

$$\pi_1(X) \cong \left\langle b, c \mid b^2c^2 = 1 \right\rangle.$$

What group is this?

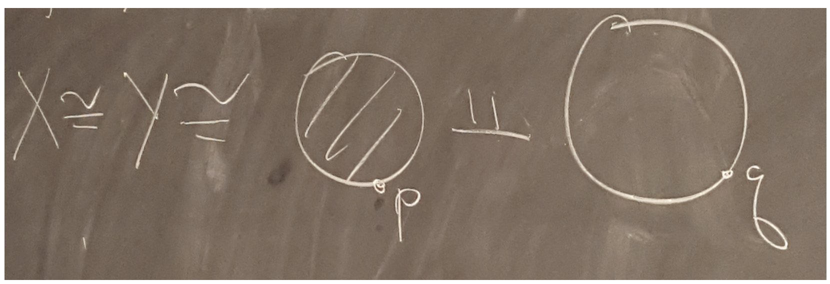
$$\begin{aligned}\mathrm{Ab}(\pi_1(X)) &= \langle b, c \mid b^2c^2 = 1, bc = cb\rangle \\ &= \langle b, c \mid (bc)^2 = 1, bc = cb\rangle \\ &= \langle bc, c \mid (bc)^2 = 1, (bc)c = c(bc)\rangle \\ &= \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}.\end{aligned}$$

So by our classification theorem for surfaces, $X \cong \mathbb{R}P^2 \# \mathbb{R}P^2$.

1.3 Homotopy equivalence and the fundamental group

If $X \simeq Y$, then is $\pi_1(X, p) \cong \pi_1(Y, q)$? This only holds true in general when X and Y are path-connected. You have to make sure that p and q are on the same connected component.

Example 1.1. Here is an example where the statement does not hold. Let $X \cong Y \cong D^2 \amalg S^1$.



Then $\pi_1(X, p) \cong 1$, but $\pi_1(Y, q) \cong \mathbb{Z}$.

However, taking care with the basepoints, we do have the following theorem.

Theorem 1.2. *If $f : X \to Y$ is a homotopy equivalence, then $\pi_1(X, p) \cong \pi_1(Y, f(p))$.*

1.4 Covering spaces and orbit spaces

Here is Problem 3b from the 2016 midterm: "Give a covering space of $\mathbb{R}P^n$."

The easiest answer to give is $\mathbb{R}P^n$ itself because $X \xrightarrow{\mathrm{id}_X} X$ is a covering map. We could also have $\mathbb{R}P^n \amalg \cdots \amalg \mathbb{R}P^n$.

If we want a nontrivial, path-connected covering space, we should use S^n. There is an action of $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ on S^n given by

$$f_0 = \mathrm{id}_{S^n}, \qquad f_1(x) = -x.$$

Then $\mathbb{R}P^n \cong S^n/(\mathbb{Z}/2\mathbb{Z})$ under this action. To show that the action is nice, take the U to be the interior of a hemisphere (say, the upper hemisphere) containing x; then $f_1(U)$ is the lower hemisphere, which is disjoint.

We also had the following theorem to help us figure out the fundamental group of $\mathbb{R}P^n$.

Theorem 1.3. *If $\pi_1(X) \cong 1$ and G acts nicely on X, then $\pi_1(X/G) \cong G$.*

Here some of the orbit spaces we talked about:

$$\mathbb{R}P^n \cong S^n/(\mathbb{Z}/2\mathbb{Z}), \qquad T^n \cong \mathbb{R}^n/\mathbb{Z}^n$$

For the midterm, you should also know about $B^n = D^n$, S^n, $\mathbb{R}^n$, the surfaces S_g, and N_g, the Klein bottle, and the Möbius strip.

Why doesn't the Möbius strip M deformation retract onto $\partial M \cong S^1$? $\pi_1(M) \cong \mathbb{Z}$, and $\pi_1(\partial M) \cong \pi_1(S^1) \cong \mathbb{Z}$. However, if $i : \partial M \to M$ were a homotopy equivalence, then $i_* : \pi_1(\partial M) \to \pi_1(M)$ is an isomorphism. Check that i_* is multiplication by 2 (or -2) and therefore can't be an isomorphism.